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Initial-Value Adjusting Method for the Solution of Nonlinear Multipoint Boundary-Value Problems

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1. INTRODUCTION

One of the most common and difficult problems in applied mathematics, nonlinear oscillations and optimal control theory, etc., are the multipoint boundary-value problems (MPBVP's) for nonlinear ordinary differential equations [1–5].

Since it is usually impossible to obtain the analytic solutions to the nonlinear MPBVP's, these problems must be solved by some numerical methods on a computer, and a number of papers have been devoted to the numerical solutions of the nonlinear MPBVP's [6–9].

The quasilinearization technique introduced by Bellman and Kalaba [10] is an effective computational technique for the solution of nonlinear boundary-value problems, which deals with the linear approximation to the original nonlinear differential equations. However, the technique possesses some disadvantages: (i) in advance, the partial derivatives of the system equation (the Jacobian matrix) must be calculated analytically, (ii) selection of the approximate solution which leads to convergence is in general difficult, (iii) the values of the system variables obtained in the previous iteration procedure must be stored. The modified quasilinearization technique developed by Baird [7] is a modification of the original quasilinearization technique and has the advantage of essentially reduced computer storage requirements. However, the technique necessitates the solution of an additional linear boundary-value problem. While, the method of adjoints proposed by Goodman and Lance [11, 12] requires, in addition to the original system of equations, a set of variational equations and the adjoint equations associated with the variational equations. Therefore, the method is inapplicable to the problems of high complexity and dimension.

In the present paper, a new iterative technique for the MPBVP's, termed initial-value adjusting method [13], is developed in order to overcome these difficulties mentioned above. The present technique only requires to solve iteratively the original nonlinear differential equations with a set of initial conditions prescribed. By calculating the differences between the values of the

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nonlinear boundary conditions obtained in the previous iteration and these obtained from the solutions of the original equations subject to the previous initial conditions with slight perturbations, the missing initial conditions are determined numerically. The necessities of determining analytically the partial derivatives of the original equations and the boundary conditions are eliminated.

Because of its relationship to the quasilinearization technique, the present technique is expected to have a nearly quadratic convergence property [16], with the advantage of essentially reduced computer storage requirements, and can be easily programmed for digital computers by usual integration routines, such as Runge–Kutta–Gill method [15].

Salient features of the present technique for the nonlinear MPBVP's are illustrated through an example.

2. Two-Point Boundary-Value Problems

In this section we consider the two-point boundary-value problems (TPBVP's) in which, for a set of ordinary differential equations, a set of linear boundary conditions is specified.

2.1. *Linear* TPBVP's

Let us first consider the n-dimensional linear differential equation

$$\dot{x} = A(t)\,x + h(t), \quad t_1 \leqslant t \leqslant t_2 \tag{2.1}$$

with the linear boundary condition

$$P_1 x(t_1) + P_2 x(t_2) = p, \tag{2.2}$$

where A is an $n \times n$-dimensional matrix continuous in t, h is an n-dimensional given vector function continuous in t, P_1 and P_2 are both $n \times n$-dimensional matrices, and p is an n-dimensional vector.

Let $\Phi(t, t_1)$ be the $n \times n$-dimensional transition matrix defined by

$$\dot{\Phi}(t, t_1) = A(t)\,\Phi(t, t_1), \qquad \Phi(t_1, t_1) = I_n, \tag{2.3}$$

where I_n is the $n \times n$-dimensional identity matrix. Then, the solution of (2.1) subject to an initial condition $x(t_1)$ can be written in terms of the transition matrix as

$$x(t) = \Phi(t, t_1)\,x(t_1) + v(t, t_1), \tag{2.4}$$

where

$$v(t, t_1) = \int_{t_1}^{t} \Phi(t, \tau)\,h(\tau)\,d\tau. \tag{2.5}$$

Let w_0 be an initial approximation to the initial condition $x(t_1)$. Then, from (2.4) with $x(t_1) = w_0$, the corresponding terminal value $x(t_2) = w_f$ is given by

$$w_f = \Phi(t_2, t_1)\, w_0 + v(t_2, t_1). \tag{2.6}$$

Similarly, let x_0 be the exact initial condition which satisfies the given boundary condition (2.2). Then, from (2.4), we have

$$x(t_2) = \Phi(t_2, t_1)\, x_0 + v(t_2, t_1). \tag{2.7}$$

Substracting (2.6) from (2.7) gives

$$x(t_2) = w_f + \Phi(t_2, t_1)\,(x_0 - w_0). \tag{2.8}$$

Now, we deduce from (2.2) and (2.8) that

$$[P_1 + P_2\Phi(t_2, t_1)]\, x_0 = p - P_2[w_f - \Phi(t_2, t_1)\, w_0]. \tag{2.9}$$

We now summarize the above discussion by presenting it in the form of a theorem.

THEOREM 2.1. *Let w_f be the terminal value of x at $t = t_2$ obtained from* (2.4) *with an arbitraty initial condition w_0. If the matrix $[P_1 + P_2\Phi(t_2, t_1)]$ is nonsingular, then, from* (2.9), *the exact initial condition x_0 of* (2.1) *which satisfies the given boundary condition* (2.2) *is given by*

$$x(t_1) = x_0 = [P_1 + P_2\Phi(t_2, t_1)]^{-1}\{p - P_2[w_f - \Phi(t_2, t_1)\, w_0]\}. \tag{2.10}$$

The theorem plays an important role in the subsequent discussions.

2.2. *Nonlinear Final-Value Problems*

Consider now the n-dimensional nonlinear differential equation

$$\dot{x} = f(t, x), \qquad t_1 \leqslant t \leqslant t_2 \tag{2.11}$$

with the given terminal condition $x(t_2) = x_f$, where it is desired to determine the unknown initial condition $x(t_1)$. The n-dimensional vector function f is twice differentiable with respect to x and continuous in t.

Suppose that $x(t) = {}^k x(t)$ is the solution of (2.11) with an arbitrary initial condition ${}^k x(t_1) = {}^k w_0$, i.e.,

$${}^k\dot{x} = f(t, {}^k x), \qquad {}^k x(t_1) = {}^k w_0 = {}^k(w_1, w_2, \ldots, w_n)', \tag{2.12}$$

where ${}^k x$ denotes x at kth iteration ($k = 0, 1, 2, \ldots$). Then the following theorem is useful.

THEOREM 2.2. *Let $A(t)$ be an arbitrary $n \times n$-dimensional matrix continuous in $t \in [t_1, t_2]$, and ${}^k z$ be the solution of the n-dimensional linear differential equation given by*

$$ {}^k\dot{z} = A(t)({}^k z - {}^k x) + f(t, {}^k x), \qquad {}^k z(t_1) = {}^k w_0 \qquad (k = 0, 1, 2, \ldots). \tag{2.13} $$

Then, (i) *the solution ${}^k z(t)$, independent of the choice of $A(t)$, coincides with ${}^k x(t)$, $\forall t \in [t_1, t_2]$, and* (ii) *if the transition matrix $\Phi(t_2, t_1)$ which satisfies the similar relation to* (2.3) *is nonsingular, then the exact initial condition ${}^{k+1}w_0$ of* (2.13) *which satisfies the given terminal condition x_f is given by*

$$ \begin{aligned} {}^{k+1}w_0 &= {}^k w_0 + \Phi^{-1}(t_2, t_1)[x_f - {}^k z(t_2)] \\ &= {}^k x(t_1) + \Phi^{-1}(t_2, t_1)[x_f - {}^k x(t_2)]. \end{aligned} \tag{2.14} $$

Proof. Letting $u(t) = {}^k z(t) - {}^k x(t)$, from (2.12) and (2.13), we have

$$ \dot{u} = A(t)\, u, \qquad u(t_1) = 0, \tag{2.15} $$

which, from the property of simultaneous differential equations, implies that $u(t) = 0$, i.e., ${}^k z(t) = {}^k x(t)$, $\forall t \in [t_1, t_2]$.

By analogy with (2.10) in Theorem 2.1, the relation (2.14) can be easily shown. Q.E.D.

From the expression (2.13), we can see that if we were allowed to select the Jacobian matrix, then the choice

$$ A(t) = \frac{\partial f(t, {}^k x)}{\partial x} \tag{2.16} $$

would yield one of the best possible convergence rates. Suppose that ${}^k\Psi(t, t_1)$ is the transition matrix satisfying the following equation:

$$ {}^k\dot{\Psi} = \left[\frac{\partial f(t, {}^k x)}{\partial x}\right] {}^k\Psi, \qquad {}^k\Psi(t_1, t_1) = I_n . \tag{2.17} $$

Let $\Phi \equiv {}^k\Psi$, and ${}^{k+1}w_0$ obtained from (2.14) be the initial condition for solving the final-value problem. Then the iteration method based on the particular choice (2.16) is reduced to the modified quasilinearization algorithm. Under appropriate conditions the algorithm has both quadratic and monotonic convergence properties [7]. However the algorithm has a disadvantage that the Jacobian matrix (2.16) must be analytically calculated in advance. Moreover, since (2.12) and (2.17) must be solved simultaneously, the algorithm is often inapplicable for problems of high complexity and dimension.

In the direct shooting method proposed by Mataušek [8], the transition matrix in (2.14) is set equal to the $n \times n$-dimensional identity matrix, i.e.,

$\Phi(t_2, t_1) = I_n$. Hence the algorithm is simple. However it is linearly convergent, so that a large number of iterations may be necessary to obtain the converged solution.

2.3. *Initial-Value Adjusting Method*

In order to overcome these disadvantages mentioned above, let us now consider an another choice for the matrix $\Phi(t_2, t_1)$. By analogy with (2.12), consider the following n-dimensional initial-value problem:

$$ {}^k\dot{y}_j = f(t, {}^k y_j), \qquad {}^k y_j(t_1) = {}^k(w_1, \ldots, w_j + \epsilon_j, \ldots, w_n)', \tag{2.18} $$

where

$$ \epsilon_j = \max[\epsilon, |\, \epsilon^k w_j \,|] \qquad (j = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots), \tag{2.19} $$

and ϵ is a small parameter, called the perturbation parameter, such that $0 < \epsilon \ll 1$. Since ϵ_j is assumed to be small enough, the solution of (2.18) may be expressed as

$$ {}^k y_j = {}^k x + \epsilon_j \delta x_j . \tag{2.20} $$

Let us now define the $n \times n$-dimensional matrix ${}^k\Psi(t, t_1; \epsilon)$ whose jth column vector ${}^k\Psi_j(t, t_1; \epsilon)$ is identical to δx_j, i.e.,

$$ {}^k\Psi_j(t, t_1; \epsilon) = \frac{1}{\epsilon_j}[{}^k y_j(t) - {}^k x(t)] \qquad (j = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots). \tag{2.21} $$

Then we have the following theorem [13].

THEOREM 2.3. *Let* ${}^k\Psi(t_2, t_1)$ *and* ${}^k\Psi(t_2, t_1; \epsilon)$ *be the* $n \times n$*-dimensional matrices defined by* (2.17) *and* (2.21), *respectively. Then the following relation holds:*

$$ \lim_{\epsilon \to 0} {}^k\Psi(t_2, t_1; \epsilon) = \lim_{\epsilon_j \to 0}[{}^k\Psi_j(t_2, t_1; \epsilon)] = {}^k\Psi(t_2, t_1). \tag{2.22} $$

Proof. Substituting (2.20) into (2.18) yields

$$ {}^k\dot{x} + \epsilon_j \delta\dot{x}_j = f(t, {}^k x + \epsilon_j \delta x_j) \qquad (j = 1, 2, \ldots, n). \tag{2.23} $$

Expanding the right side of (2.23) in a Taylor's series reduces (2.23) to

$$ {}^k\dot{x} + \epsilon_j \delta\dot{x}_j = f(t, {}^k x) + \epsilon_j \left[\frac{\partial f(t, {}^k x)}{\partial x}\right] \delta x_j + O(\epsilon_j^2), \tag{2.24} $$

where $O(\epsilon_j^2)$ denotes the high-order infinitesimal terms. Subtracting (2.12) from (2.24), we have

$$ \delta\dot{x}_j = \left[\frac{\partial f(t, {}^k x)}{\partial x}\right] \delta x_j + \frac{1}{\epsilon_j} O(\epsilon_j^2), \qquad \delta x_j(t_1) = e_j , \tag{2.25} $$

where $e_j = (0,..., 1,..., 0)'$ ($j = 1, 2,..., n$) is the jth unit vector. As ϵ approaches zero, $O(\epsilon_j^2)/\epsilon_j$ tends to zero, and thus, in view of (2.17), (2.22) is established. Q.E.D.

By replacing $\Phi(t_2, t_1)$ in (2.14) by ${}^k\Psi(t_2, t_1; \epsilon)$ obtained from (2.21), we now have a new iteration algorithm given by

$$ {}^{k+1}x_0 = {}^kx_0 + {}^k\Psi^{-1}(t_2, t_1; \epsilon)\,[x_f - {}^kx(t_2)] \qquad (k = 0, 1, 2,...). \tag{2.26} $$

The present algorithm given by (2.21) and (2.26) is termed here "initial-value adjusting method".

3. Nonlinear Multipoint Boundary-Value Problems

Let us now consider the nonlinear multipoint boundary-value problems using the result of the previous section. The system under consideration is governed by the n-dimensional nonlinear ordinary differential equation:

$$ \dot{x} = f(t, x), \qquad x = (x_1, x_2,..., x_n)', \tag{3.1} $$

and the nonlinear boundary condition is given by

$$ g(x(t_1), x(t_2),..., x(t_m)) = 0 \qquad t_1 < t_2 < \cdots < t_m, \tag{3.2} $$

where the n-dimensional vector function f is twice differentiable with respect to x and continuous in t, and the n-dimensional vector function g is twice differentiable with respect to $x(t_i)$ ($i = 1, 2,..., m$; $m \geqslant 2$).

3.1. *Quasilinearization Technique for the* MPBVP's

As well-known, the quasilinearization technique is a method to solve a finite set of nonlinear equations by solving a sequence of linear equations. In this subsection an iterative algorithm with the technique is developed to solve the MPBVP [12].

Let us again consider the n-dimensional linear differential equation (2.1), i.e.,

$$ \dot{x} = A(t)\,x + h(t), \qquad t_1 \leqslant t \leqslant t_m \quad (m \geqslant 2) \tag{3.3} $$

with the nonlinear boundary condition (3.2). It follows, from the definition of the transition matrix $\Phi(t_i, t_1)$ given by (2.3), that the solution at $t = t_i$ ($i = 2, 3,..., m$) of (3.3) with the initial condition ${}^{k+1}x(t_1)$ is given by

$$ {}^{k+1}x(t_i) = \Phi(t_i, t_1)\,{}^{k+1}x(t_1) + v(t_i, t_1), \tag{3.4} $$

where $v(t_i, t_1)$ is given by (2.5).

Let us replace (3.2) by

$$^{k+1}g = g(^{k+1}x(t_1), ^{k+1}x(t_2),..., ^{k+1}x(t_m)) = 0, \tag{3.5}$$

and expand the right side of (3.5) in a Taylor's series up through first-order terms around the nominal solution $^{k}x(t_i)$ $(i = 1, 2,..., m)$. Then, we have

$$^{k+1}g = {}^{k}g + \sum_{i=1}^{m} \left[\frac{\partial {}^{k}g}{\partial x(t_i)}\right] [{}^{k+1}x(t_i) - {}^{k}x(t_i)] = 0. \tag{3.6}$$

Substituting (3.4) into (3.6) yields

$$^{k}S[^{k+1}x(t_1) - {}^{k}x(t_1)] = -{}^{k}g, \tag{3.7}$$

where

$$^{k}S = \sum_{i=1}^{m} \left[\frac{\partial {}^{k}g}{\partial x(t_i)}\right] \Phi(t_i, t_1). \tag{3.8}$$

From the above discussion, we have the following theorem.

THEOREM 3.1. *If the $n \times n$-dimensional matrix ^{k}S is nonsingular, then the new initial condition $^{k+1}x(t_1)$ of* (3.3) *which satisfies the linearized boundary condition* (3.6) *is given by*

$$^{k+1}x(t_1) = {}^{k}x(t_1) - {}^{k}S^{-1}{}^{k}g \qquad (k = 0, 1, 2,...). \tag{3.9}$$

All the terms on the right side of (3.8) are known, hence the solution of (3.9) gives the complete set of initial conditions sufficient for the numerical integration of (3.3). However the technique requires that (i) the partial derivative $\partial g/\partial x(t_i)$ $(i = 1, 2,..., m)$ must be analytically determined, (ii) the transition matrix $\Phi(t_i, t_1)$ $(i = 1, 2,..., m)$ must be computed and stored. For the nonlinear MPBVP's of high complexity and dimension, the problem preparations may be arduous and often impractical.

3.2. *Initial-Value Adjusting Method for the* MPVBP's

We now examine the extension of the initial-value adjusting method to the MPBVP given by (3.2) and (3.3).

By analogy with (2.18), let us define the n-dimensional perturbed initial condition $^{k}y_j(t_1)$ by

$$^{k}y_j(t_1) = {}^{k}x(t_1) + \epsilon_j e_j \qquad (j = 1, 2,..., n), \tag{3.10}$$

where ϵ_j is defined by (2.19) and e_j is the jth unit vector. Then, in view of the analogy with (3.4), the corresponding solution at $t = t_i$ is given by

$$^{k}y_j(t_i) = {}^{k}\Psi(t_i, t_1; \epsilon)\, {}^{k}y_j(t_1) + v(t_i, t_1; \epsilon) \qquad (j = 1, 2,..., n;\ i = 2, 3,..., m), \tag{3.11}$$

where ${}^k\Psi(t_i, t_1; \epsilon)$ is defined by (2.21), and $v(t_i, t_1; \epsilon)$ is identical to (2.5) with ${}^k\Psi(t, \tau; \epsilon)$.

Substituting (3.11) into (3.2), we then obtain

$$\begin{aligned} {}^kg_j &= {}^kg(y_j(t_1), y_j(t_2),..., y_j(t_m)) \\ &= g({}^ky_j(t_1), {}^k\Psi(t_2, t_1; \epsilon)\, {}^ky_j(t_1) + v(t_2, t_1; \epsilon),..., {}^k\Psi(t_m, t_1; \epsilon)\, {}^ky_j(t_1) \\ &\quad + v(t_m, t_1; \epsilon)) = 0. \end{aligned} \tag{3.12}$$

On expanding the right side of (3.12) in Taylor's series around nominal solution ${}^kx(t_1)$ and then subtracting ${}^kg$ from the resulting solution, we have

$$\frac{1}{\epsilon_j}({}^kg_j - {}^kg) = {}^kS(\epsilon)\, e_j, \qquad (j = 1, 2,..., n), \tag{3.13}$$

where

$${}^kS(\epsilon) = \sum_{i=1}^{m} \left[\frac{\partial {}^kg}{\partial x(t_i)}\right] \Phi(t_i, t_1) + \left[\frac{1}{\epsilon_j} O(\epsilon_j{}^2)\right]. \tag{3.14}$$

The second term in the right side of (3.14) denotes the high-order infinitesimal term whose jth column vector is identical to $O(\epsilon_j{}^2)/\epsilon_j$ ($j = 1, 2,... n$). We now have the following theorem for the initial-value adjusting method.

THEOREM 3.2. *Let ${}^kS$ and ${}^kS(\epsilon)$ be the $n \times n$-dimensional matrices defined by* (3.8) *and* (3.14), *respectively. Then the following relation holds:*

$$\lim_{\epsilon \to 0} {}^kS(\epsilon) = \lim_{\epsilon_j \to 0} \left[\frac{1}{\epsilon_j}({}^kg_j - {}^kg)\right] = {}^kS \qquad (j = 1, 2,..., n). \tag{3.15}$$

Proof. By analogy with Theorem 2.3, $O(\epsilon_j{}^2)/\epsilon_j$ is a term with the property that

$$\lim_{\epsilon_j \to 0} \frac{1}{\epsilon_j} O(\epsilon_j{}^2) = 0 \qquad (j = 1, 2,..., n). \tag{3.16}$$

On taking into account the fact that (3.3) is linear, from Theorem 2.3, we have

$$\lim_{\epsilon \to 0} {}^k\Psi(t_i, t_1; \epsilon) = \Phi(t_i, t_1) \qquad (i = 2, 3,..., m;\ k = 0, 1, 2,...), \tag{3.17}$$

which, in turn, implies that ${}^kS(\epsilon)$ tends to ${}^kS$ as $\epsilon \to 0$. Q.E.D.

If the $n \times n$-dimensional matrix ${}^kS(\epsilon)$ given by (3.14) is nonsingular, then, with the analogy to (3.9), we have a new iteration algorithm defined by

$${}^{k+1}x(t_1) = {}^kx(t_1) - {}^{k+1}\alpha[{}^kS(\epsilon)]^{-1}\, {}^kg \qquad (k = 0, 1, 2,...), \tag{3.18}$$

where ${}^{k+1}\alpha$ is the relaxation factor such that $0 < {}^{k+1}\alpha \leqslant 1$ ($k = 0, 1, 2,...$). If

${}^{k+1}\alpha = 1$, the full correction is taken. However, for the problem which is very sensitive to the initial conditions, the full correction may be too much. The present algorithm given by (3.14) and (3.18) is termed here "initial-value adjusting method for the nonlinear MPBVP's".

3.3. *Computational Algorithm for the Initial-Value Adjusting Method*

From the results of Subsections 2.3 and 3.2, the initial-value adjusting method for nonlinear MPBVP given by (3.1) and (3.2) is carried out as follows:

Step 0. Set $k = 0$, and prescribe the values of the initial condition ${}^0x(t_1)$, the perturbation parameter ϵ, the relaxation factor ${}^1\alpha$, and the convergence criterion σ ($\sigma > 0$).

Step 1. Compute the initial-value problem (3.1) with the initial condition $x(t_1) = {}^kx(t_1)$, and obtain the resulting value of the nonlinear boundary conditions (3.2), i.e.,

$$ {}^kg = g({}^kx(t_1), {}^kx(t_2),..., {}^kx(t_m)). \tag{3.19} $$

Step 2. Compute the error ${}^kG$ defined by

$$ {}^kG = \left(\frac{1}{n}\, {}^kg'\, {}^kg\right)^{1/2}. \tag{3.20} $$

If ${}^kG \leqslant \sigma$, then terminate the procedure. If ${}^kG > \sigma$ and ${}^kG < 0.1^{k-1}G$, then set ${}^{k+1}\alpha = \min[1.2^k\alpha, 1]$, and if ${}^kG > \sigma$ and ${}^kG \geqslant {}^{k-1}G$, then set ${}^{k+1}\alpha = 0.8^k\alpha$, and proceed to the next step.

Step 3. Set $j = 1$.

Step 4. Compute the initial-value problem (3.1) with the perturbed initial condition $x(t_1) = {}^ky_j(t_1)$, where ${}^ky_j(t_1)$ is given by (3.10). Obtain the resulting value ${}^kg_j$ given by

$$ {}^kg_j = g({}^ky_j(t_1), {}^ky_j(t_2),..., {}^ky_j(t_m)). \tag{3.21} $$

Then determine the matrix ${}^kS(\epsilon)$ whose jth column vector ${}^kS_j(\epsilon)$ is given by

$$ {}^kS_j(\epsilon) = \frac{1}{\epsilon_j}\,({}^kg_j - {}^kg). \tag{3.22} $$

Step 5. If $j < n$, then set $j = j + 1$ and return to Step 4. Otherwise proceed to the next step.

Step 6. From (3.18) with (3.22), determine the initial condition ${}^{k+1}x(t_1)$ for the next iteration. Then set $k = k + 1$, and return to Step 1.

In order to compute corrections to the set of missing initial conditions, the initial-value adjusting method requires only $(n + 1)$ integrations of the original nonlinear differential equation (3.1) at each iteration. The matrix ${}^kS(\epsilon)$ is

directly calculated from (3.22). Consequently the necessities of determining analytically the partial derivatives $\partial f/\partial x$ and $\partial g/\partial x(t_i)$ and computing the transition matrix ${}^k\Psi(t_i, t_1; \epsilon)$ $(i = 2, 3, \ldots, m)$ are eliminated. Moreover, since the initial condition ${}^kx(t_1)$ and the value ${}^kg$ of the boundary condition are the only quantities to be stored, the present method yields significant savings in computer storage.

4. Illustrative Example

In this section, the orbital intercept example of McGill and Kenneth [14] is examined to illustrate the application of the present method.

The sixth-order nonlinear differential equations of the example are given by

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_3 = x_4, \qquad \dot{x}_5 = x_6, \\ &\dot{x}_2 = -\frac{x_1}{r^3}, \qquad \dot{x}_4 = -\frac{x_3}{r^3}, \qquad \dot{x}_6 = -\frac{x_5}{r^3}, \end{aligned} \tag{4.1}$$

where

$$r = [x_1^2 + x_3^2 + x_5^2]^{1/2}. \tag{4.2}$$

The nonlinear boundary conditions are taken to be

$$\begin{aligned} &g(1) = x_1(0) - 1.076 = 0, \\ &g(2) = x_2(0) + x_2(0)^2 + x_2(0.8) + x_2(1.4) + x_2(2) + 2.053292953504164 = 0, \\ &g(3) = x_3(0) + x_4(0) - 0.472283099142472 = 0, \\ &g(4) = x_4(0.8)^2 + x_5(1.4) - 1.040204804411078 = 0, \\ &g(5) = x_3(2) + x_5(2) - 1.57661000000000 = 0, \\ &g(6) = x_6(2) + 0.03407297218269353 = 0, \end{aligned} \tag{4.3}$$

where the values in (4.3) are obtained from the solutions of the TPBVP's given by (4.1) and the boundary conditions:

$$\begin{aligned} &x_1(0) = 1.076000000000000, \qquad x_1(2) = 0.000000000000000, \\ &x_3(0) = 0.000000000000000, \qquad x_3(2) = 0.576000000000000, \\ &x_5(0) = 0.000000000000000, \qquad x_5(2) = 0.997661000000000. \end{aligned} \tag{4.4}$$

For the numerical integration of the differential equations, the fourth-order Runge–Kutta–Gill method written in double precision arithmetic is adopted, where use is made of five hundred grid-points on the interval [0, 2]. In the computation, the initial relaxation factor $\alpha = 1.0$ and the convergence criterion

$\sigma = 10^{-14}$ are used. The trial initial values are set equal to the McGill and Kenneth's starting functions [14]. The values are

$$\begin{aligned} x_1(0) &= 1.076000000000000, & x_2(0) &= 0.538000000000000, \\ x_3(0) &= 0.000000000000000, & x_4(0) &= 0.288000000000000, \\ x_5(0) &= 0.000000000000000, & x_6(0) &= 0.498830000000000. \end{aligned} \tag{4.5}$$

Figure 1 shows a variation of the computer time T_c necessary for convergence with the perturbation parameter ϵ, where the numbers in paraenthesis denote the iteration numbers. It is easily seen from the figure that the convergence time is insensitive to the values of the perturbation parameter. For the values of $\epsilon = 10^{-4} \sim 10^{-11}$, it took eight iterations to attain the convergence criterion. In Table 1, we list in double precision the convergence behavior of the error ${}^kG$ defined by (3.20). Table 2 shows the missing initial values and the resulting terminal values obtained by the present algorithm.

All the numerical computations were obtained by using FACOM M-190 at the Data Processing Center of Kyoto University.

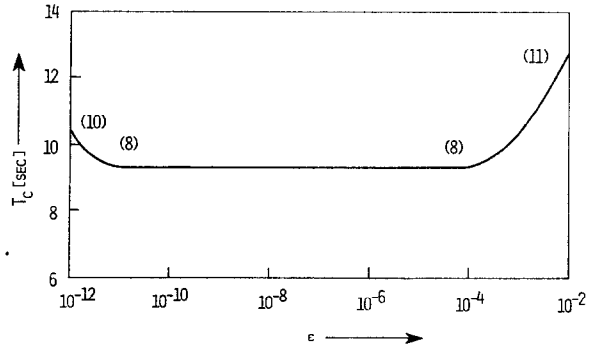

FIG. 1. A variation of the convergence time T_c with the perturbation parameter ϵ.

TABLE 1

A Convergence Behavior of the Error ${}^kG$ ($\epsilon = 10^{-10}$)

k	G	k	G
0	0.892966812870739D0	5	0.137484997318239$D-4$
1	0.249436261529752D0	6	0.179180858225079$D-7$
2	0.120983244257701$D-1$	7	0.620893731575291$D-13$
3	0.328369357867840$D-2$	8	0.223307048002705$D-16$
4	0.420007164711370$D-3$		

TABLE 2

The Initial and Terminal Values Obtained by the Initial-Value Adjusting Method ($\epsilon = 10^{-10}$)

	Initial Values ($t = 0$)	Terminal Values ($t = 2$)
x_1	0.10760000000000D1	−0.16903615377753D−15
x_2	0.10165902478857D1	−0.88225106714809D0
x_3	0.63767463420841D−15	0.57600000000000D0
x_4	0.47228309914247D0	−0.19672044890230D−1
x_5	0.54573575737601D−15	0.99766100000000D0
x_6	0.81801810585691D0	−0.34072972182694D−1

5. Concluding Remarks

An iterative algorithm, termed initial-value adjusting method, has been developed to solve nonlinear multipoint boundary-value problems for nonlinear ordinary differential equations, and justified through the solution of an example.

Some of the important advantages of the present method are summarized as follows: (i) since, at each iteration, the present method deals with directly the original differential equations and the given boundary conditions, it does require neither the solution of an additional linear equation nor the necessity of calculating analytically the partial derivatives of the differential equations and the given boundary conditions, (ii) because of its relationship to the quasilinearization technique, it is expected, for the sufficiently small perturbation parameter, to have a nearly quadratic convergence, (iii) since the initial conditions and the values of the boundary conditions at the previous iteration are the only quantities to be stored, it essentially eliminates the necessity for computer storages, (iv) it can be easily programmed for a digital computer by more or less standard integration technique such as Runge–Kutta–Gill method.

Acknowledgment

The initial-value adjusting method for the solution of nonlinear MPBVP's is catalogued in the Scientific Subroutine Library of Data Processing Center of Kyoto University, Kyoto, Japan [15]. The authors wish to acknowledge the kind considerations of the staffs of the center, who have made computer-time available to the authors.

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